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Structure of the transient wall-friction law in one-dimensional models of laminar pipe flows

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The problem of describing an unsteady cylindrical pipe flow with one-dimensional equations is investigated, and an exact method for obtaining a closure relationship is proposed for the transient shear stress in a laminar flow submitted to an arbitrary transient pressure gradient. Extensive comparisons are given for a step or a harmonic pressure gradient between the approximate solution derived from this method, some results of the literature and exact solutions of the Navier–Stokes equations.

1. Introduction

The use of one-dimensional models with cross-section-averaged flow variables simplifies considerably the analysis of pipe flows for most engineering applications. A correct method for obtaining cross-section-averaged equations for a pipe flow (which applies as well to homogeneous fluid as to two-phase flows) can be found in Delhaye & Achard (1976). For the cases where the fluid is incompressible and the pipe a circular cylinder, the momentum and energy equations take the following form:

$$\frac{\partial|w|}{\partial t}+\frac{\partial}{\partial z}(k_m|w|^2) = -\frac{1}{\rho}\frac{\partial|p|}{\partial z}+\frac{1}{\rho}\frac{\partial(\tau_v)}{\partial z}+|\mathbf{F}\,.\,\mathbf{n}_z|-\frac{2\tau}{\rho}, \tag{1.1}$$

$$\frac{\partial|T_f|}{\partial t}+\frac{\partial}{\partial z}(k_T|w|\,|T_f|) = \alpha\frac{\partial^2|T_f|}{\partial z^2}+\frac{2q}{\rho C_p}, \tag{1.2}$$

where z is the axial co-ordinate, and $|\,.\,|$ represents the averaging operator in a cross-section normal to the unit vector $\mathbf{n}_z$.

However, the drawback of such an approach lies in the need for closure relationships for the unknown terms of the averaged formulation in terms of the new state variables: the mean axial velocity $|w| = |\mathbf{v}\,.\,\mathbf{n}_z|$ and the mean temperature $|T_f|$. The first equation requires knowledge of the average viscous stress $\tau_v = |(\mathbf{n}_z\,.\boldsymbol{\tau})\,.\,\mathbf{n}_z|$, the ratio of the exact momentum flux to the averaged one $k_m = |w^2|/|w|^2$, and the wall friction $\tau = -R^{-1}(\mathbf{n}_w\,.\,\boldsymbol{\tau})\,.\,\mathbf{n}_z$. The second one requires knowledge of the ratio of the exact heat flux to the averaged one $k_T = |w\,.\,T|/|w|\,|T|$ and the wall heat flux $q = -R^{-1}\mathbf{n}_w\,.\,\mathbf{q}$.

Among these closure relationships, the two wall-exchange terms τ and q are of the utmost importance and are generally written using the assumption of quasi-steady flow:

$$\tau = \tfrac{1}{2}\rho C_f|w|^2, \quad q = h(T_w-|T_f|) \tag{1.3), (1.4}$$

and both friction factor C_f and heat-transfer coefficient h are determined from experiments where the flow is *steady* and *fully developed*. However, these exchange laws are commonly used in one-dimensional models describing transient or non-fully developed processes without paying much attention to their limitations, which yields either inaccurate results or artificially unstable computations.

The limitations of the quasi-steady assumption and of the resulting laws are in fact well known among researchers (see, for example, Nusselt 1910; Lambossy 1952; Perlmutter & Siegel 1961; Koshkin *et al.* 1970). However, a correct extension of the validity of such laws to transient cases has never been stated, except perhaps by Stein (1971). This extension involves much more than a simple refitting of the coefficients C_f and h, and any correlation of those coefficients in typical unsteady cases (step, harmonic oscillations, etc.) in terms of independent variables, time or frequency, has no further importance than pointing out that the quasi-steady wall laws cannot be applied to such cases. This is because the *structure* of the laws must be modified to allow the description of an experiment with any time evolution of the control variables.

Our purpose is to present an original method for extending the classical quasi-steady equations to transient flows. Starting from the time-dependent and local (two-dimensional with the hypothesis of a cylindrical flow) balance equations, we work out the conditions for which the averaged equations give the same result. These conditions bear precisely on the form of the unknown wall laws which are needed in the averaged equations. The proposed method involves no preliminary assumption on the structure of the laws, and it results in an exact relationship which can be solved either with a high-frequency approximation or (which is more interesting for practical purposes) with a low-frequency approximation which exhibits several relaxation coefficients.

The present paper illustrates the above method by applying it to the basic case of an isothermal laminar flow in a cylindrical tube and deriving the transient shear-stress law. The more general case of a turbulent flow could be considered as well, provided a local and analytically simple model for the mean unstationary velocity would be available.

The heat-transfer case can be treated in a similar way. Work on this subject establishing a transient heat flux law is in progress.

2. Transient wall-friction law for an isothermal laminar flow in a cylindrical tube

This type of flow has been known for a long time, and exact solutions exist in the literature for various time-dependent pressure gradients, for instance a step function at time $t = 0$ (Szymanski 1932) or a harmonic pressure gradient (Sexl 1930).

2.1. *The solution in terms of local variables*

The equation for the axial component of such a flow is obviously

$$\rho\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}+\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right), \tag{2.1}$$

where the pressure gradient $-\partial p/\partial z$ is a given function of time $G(t)$ assumed zero for $t \leqslant 0$. Hence the initial condition is

$$w = 0 \quad \text{for} \quad t = 0, \tag{2.2}$$

and the boundary conditions are

(i) no slip at the wall

$$w = 0 \quad \text{for} \quad r = R; \tag{2.3}$$

(ii) axisymmetry with a tangential momentum balance on the axis

$$\frac{\partial w}{\partial r} = 0 \quad \text{for} \quad r = 0. \tag{2.4}$$

It will be useful to work with dimensionless equations: choosing R as a reference length, R^2/ν as a reference time, and denoting reference values of axial velocity and pressure gradient by W_0 and G_0 respectively. Equation (2.1) then takes the simple form

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{\partial w}{\partial t} = -G(t) \tag{2.5}$$

provided W_0 and G_0 are related by

$$W_0 = \frac{R^2 G_0}{\mu}. \tag{2.6}$$

A straightforward method for obtaining the solution with any given $G(t)$ is to write the time Laplace transform of equation (2.5). Using s as the dimensionless Laplace variable and an asterisk to denote Laplace transforms, the above equation is transformed into a Bessel equation of first kind and order zero. Its solution is

$$w^*(r, s) = \frac{1}{s}\left[1 - \frac{I_0(rs^{\frac{1}{2}})}{I_0(s^{\frac{1}{2}})}\right] G^*(s) \tag{2.7}$$

$$\stackrel{\Delta}{=} H^*(r, s)\, G^*(s), \tag{2.8}$$

where I_0 is the modified Bessel function of first kind and order zero.

In the time domain, the solution is the convolution product

$$w(r, t) = \int_0^t H(r, t')\, G(t - t')\, dt', \tag{2.9}$$

and H, the original of H^*, can be worked out simply since the exact solution w_s for a step function is known; hence

$$H(r, t) = \frac{\partial w_s}{\partial t} \tag{2.10}$$

with

$$w_s = \tfrac{1}{4}(1 - r^2) - 2 \sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)}{\alpha_n^3 J_1(\alpha_n)}\, e^{-\alpha_n^2 t} \tag{2.11}$$

(α_n is the nth positive zero of the Bessel function J_0).

2.2. *The solution in terms of average variables*

The averaged equation, (1.1), has already been given. In a laminar flow τ_v is equal to zero since the diagonal of tensor $\boldsymbol{\tau}$ contains only zeros. Moreover in the particular case of the Poiseuille flow the term $\partial(k_m |w|^2)/\partial z$ also reduces to zero. Finally, with external forces assumed to be irrotational and their potential included in the pressure equation, (1.1) becomes, in its dimensionless form,

$$\frac{dW}{dt} = G(t) - 2\tau, \tag{2.12}$$

where W stands for $|w|$.

The initial condition is the averaged form of (2.2),

$$W = 0 \quad \text{for} \quad t = 0. \tag{2.13}$$

Expressed in terms of average variables, the partial differential problem defined by (2.1) and the conditions (2.2), (2.3) and (2.4) reduces to an ordinary differential problem, completed by the closure relationship of τ versus the only state variable W.

As the solution of the averaged problem $W(t)$ must be the average of the solution of the local problem $|w(r,t)|$, we obtain after (2.9)

$$W(t) = \int_0^t |H(r,t')|\, G(t-t')\,dt' \tag{2.14}$$

and, since $\tau(t) = -(\partial w/\partial r)_{r=1}$ is also related to G,

$$\tau(t) = -\int_0^t \frac{\partial}{\partial r}[H(r,t')]_{r=1}\, G(t-t')\,dt'. \tag{2.15}$$

Thus, we observe that the two time functions $W(t)$ and $\tau(t)$ cannot be independent, since they are generated by a single forcing function $G(t)$, and that the elimination of $G(t)$ between (2.14) and (2.15) will result in the relationship sought, $\tau(W)$. This elimination can easily be achieved in the image space of Laplace, since convolution products are transformed into algebraic relations. We obtain

$$W^*_{(s)} = |H^*|\, G^*(s) = \frac{1}{s}\frac{I_2(s^{\frac{1}{2}})}{I_0(s^{\frac{1}{2}})}\, G^*(s), \tag{2.16}$$

$$\tau^*(s) = -\frac{\partial}{\partial r}[H^*(r,s)]_{r=1}\, G^*(s) = \frac{s^{\frac{1}{2}} I_1(s^{\frac{1}{2}})}{s I_0(s^{\frac{1}{2}})}\, G^*(s) \tag{2.17}$$

and after elimination of $G^*(s)$

$$\tau^*(s) = \frac{I_1(s^{\frac{1}{2}})}{s^{\frac{1}{2}} I_2(s^{\frac{1}{2}})}\, s W^*(s). \tag{2.18}$$

We note that this result has already been obtained by Zielke (1968) although written in a less convenient form. However, he did not obtain from (2.18) all the practically applicable results.

It is easy to verify that, if we use equation (2.18) when solving the averaged problem defined by (2.12) and (2.13) after a Laplace transform, we recover (2.16). Hence, the original of (2.18) is actually the exact relationship between τ and W.

Before coming back to the originals, let us give a physical interpretation of (2.18) with the help of well-known concepts of control theory. In (2.18), $2\tau^*$ appears to result from sW^* through the use of a transfer function

$$H^*_\tau(s) = \frac{2 I_1(s^{\frac{1}{2}})}{s^{\frac{1}{2}} I_2(s^{\frac{1}{2}})}. \tag{2.19}$$

So, if we insert

$$2\tau^* = H^*_\tau\, s W^* \tag{2.20}$$

into the Laplace-transformed equation (2.12), we obtain an alternative form of (2.16),

$$W^*(s) = \left[\frac{1}{1+H^*_\tau(s)}\right]\left[\frac{1}{s}\right] G^*(s), \tag{2.21}$$

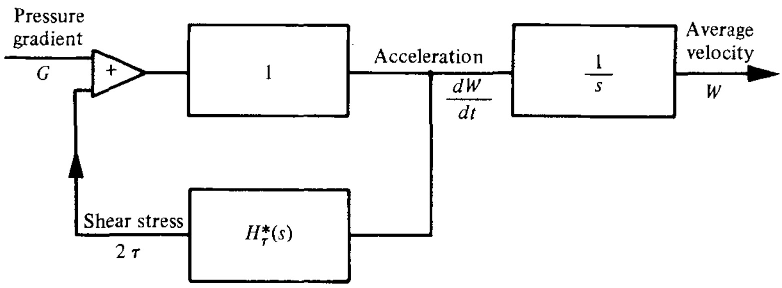

FIGURE 1. Schematic diagram illustrating the feedback effect of viscosity.

which can be represented by the schematic diagram of figure 1. $H_\tau^*(s)$ appears in a feedback loop: this agrees with the regulation effect usually assigned to viscosity.

The original of equation (2.20) is a convolution product

$$\tau(t) = \frac{1}{2}\int_0^t H_\tau(t-t')\frac{dW}{dt'}\,dt' \tag{2.22}$$

in which H_τ of H_τ^* can be obtained through the complex inversion formula and the calculus of residues:

$$H_\tau(t) = L^{-1}[H_\tau^*(s)] = 8+4\sum_{n=1}^{\infty} e^{-\gamma_n^2 t} \tag{2.23}$$

(γ_n is the nth positive zero of J_2).

Finally, the resulting relationship between $\tau(t)$ and $W(t)$ can be written

$$\tau(t) = 4W+2\sum_{n=1}^{\infty}\int_0^t \exp[-\gamma_n^2(t-t')]\frac{dW}{dt'}\,dt', \tag{2.24}$$

but it is much too cumbersome to be useful for practical applications, so approximations are needed even if only valid in limited frequency ranges.

2.3. *Modal approximation*

The Laplace transform of (2.24) is a series involving the γ_n:

$$\tau^*_{(s)} = 4W^*+2W^*s\sum_{n=1}^{\infty}\frac{1}{s+\gamma_n^2}. \tag{2.25}$$

The first approximation, when $s \to 0$ (i.e. when $t \to \infty$ or for very slowly varying motions), is obviously the quasi-steady approximation

$$\tau = 4W, \tag{2.26}$$

from which the classical value of the friction factor $C_f = 16/Re$ can be recovered.

Higher-order approximations can be obtained by truncation of the series at any order $N \geqslant 1$ and, the faster the transients, the higher N must be. From a general point of view, it is obvious that, for a chosen value of N, and after multiplying both members of (2.25) by the appropriate denominator, s^N will appear as a factor of τ^* and W^*: this means that, in the time domain, an Nth-order approximation of (2.25)

$N \backslash k$	1	2	3
1	$3{\cdot}8\times10^{-2}$	—	—
2	$5{\cdot}2\times10^{-2}$	$5{\cdot}4\times10^{-4}$	—
3	$5{\cdot}9\times10^{-2}$	$9{\cdot}2\times10^{-4}$	$4{\cdot}0\times10^{-6}$

TABLE 1

gives an Nth-order differential equation. After a recurrence calculation of the coefficients, this equation can be written

$$\tau+\sum_{k=1}^{N}\sigma_k^N\frac{d^k\tau}{dt^k}=4W+2\sum_{k=1}^{N}(2+k)\sigma_k^N\frac{d^kW}{dt^k}, \tag{2.27}$$

where σ_k^N is defined by

$$\sigma_k^N \overset{\Delta}{=} \sum_{n_1,n_2,\ldots,n_k}^{N}\frac{1}{\gamma_{n_1}^2\gamma_{n_2}^2\cdots\gamma_{n_k}^2}, \tag{2.28}$$

which means a summation performed over k indices combined without repetition among the N integer values 1 to N.

Equation (3.27), together with the initial one-dimensional equation (3.12), gives a differential system in τ and W which can be solved for any transient process defined by a given $G(t)$: this being valid at any order N, we can state that equations (3.12) and (3.27) give the general solution of any transient laminar flow in a cylindrical pipe.

The major practical problem is obviously the computation of the numerical values of the coefficients σ_k^N from the values of γ_n. These values can be found in Abramowitz & Stegun (1970) with an accuracy of 10^{-5}. The first three approximations are listed in table 1. We notice

(i) that the order of magnitude of σ_k^N decreases rapidly when k increases. This confirms the expected strong influence of the very first derivatives and shows that the truncation at small orders (e.g. 2 or 3) would generally be sufficient.

(ii) That for a given order N of the differential equation, the kth coefficient is a function of N. This is hardly acceptable, and an ideal approximation would be to limit the influence of truncation to the order of the differential equation only. This can be achieved by replacing σ_k^N by its limit σ_k when $N\to\infty$, and the identification between the series expansion of $I_2(s^{\frac{1}{2}})$ and the infinite product expansion of $J_2(is^{\frac{1}{2}}) = -I_2(s^{\frac{1}{2}})$ provides a general expression for σ_k, *viz.*

$$\sigma_k=\frac{2}{4^k k!\,(k+2)!}. \tag{2.28'}$$

The first values of k given by (2.28′) are shown in table 2.

k	1	2	3
σ_k	$8{\cdot}33\times10^{-2}$	$2{\cdot}60\times10^{-3}$	$4{\cdot}34\times10^{-5}$

TABLE 2

The order of magnitude of σ_1 is 10^{-1}; so the first-order approximation of (2.25),

$$\tau+\sigma_1\frac{d\tau}{dt}=4W+6\sigma_1\frac{dW}{dt}, \tag{2.29}$$

applied to a periodic flow, is expected to bring relevant corrections to the quasi-steady approximation (3.26) for values of the dimensionless frequency within the approximate range [0, 10]. In the same way, the order of magnitude of σ_2 is 10^{-3}, and so the second-order approximation of (2.25),

$$\tau+\sigma_1\frac{d\tau}{dt}+\sigma_2\frac{d^2\tau}{dt^2} = 4W+6\sigma_1\frac{dW}{dt}+8\sigma_2\frac{d^2W}{dt^2}, \tag{2.30}$$

will introduce sufficient corrections up to $\omega \sim 10^{\frac{3}{2}}$, etc.

Further computations will confirm these orders of magnitude and, in addition, show that the next approximations are not worth considering. Let us write the transfer function $H^*(s)$ which relates the average velocity $W^*(s)$ to the pressure gradient $G^*(s)$, using the three successive approximations (2.26), (2.29) and (2.30):

$$H_0^*(s) = \frac{1}{8+s}, \tag{2.31}$$

$$H_1^*(s) = \frac{1+\sigma_1 s}{8+(1+12\sigma_1)s+\sigma_1 s^2}, \tag{2.32}$$

$$H_2^*(s) = \frac{1+\sigma_1 s+\sigma_2 s^2}{8+(1+12\sigma_1)s+(\sigma_1+16\sigma_2)s^2+\sigma_2 s^2}. \tag{2.33}$$

We compare, within the range $\omega \in [10^{-1}, 10^4]$:

(i) the exact amplitude characteristic of the frequency response $|H(i\omega)|$ with the approximations $|H_N^*(i\omega)|$ ($N = 0, 1, 2$; see figure 2*a*);

(ii) the exact phase characteristic $\arg[H^*(i\omega)]$ with the approximation $\arg[H_N^*(i\omega)]$ (see figure 2*b*);

(iii) the relative errors

$$\epsilon_{aN} = \frac{|H_N^*(i\omega)|-|H^*(i\omega)|}{|H^*(i\omega)|}$$

for amplitudes (see figure 3*a*) and

$$\epsilon_{pN} = \frac{\arg[H_N^*(i\omega)]-\arg[H^*(i\omega)]}{\arg[H^*(i\omega)]}$$

for phases (see figure 3*b*).

From the above results, we observe that the relative errors tend to zero at low frequencies, and that they become less than 5 % for maximum values of ω given in table 3, which confirm very closely the values expected from the orders of magnitude of the coefficients $\tilde{\sigma}_k$.

Another observation may be more surprising, since the modal approximation is known to be valid at low frequencies: the relative errors also tend to zero for large values of ω because, despite very poor modelling of viscous effects, these effects themselves become negligible compared with inertial effects.

Thus, we can come to the conclusion that the shear stress law obtained in (2.27) gives good results, except for an intermediate gap, in the whole range of frequencies. The lower limit of this interval increases with N, whereas the upper limit is almost insensitive to N, and the maximum error decreases when N increases. So, for any given value of ϵ, it is theoretically possible to reduce this gap to zero by increasing N.

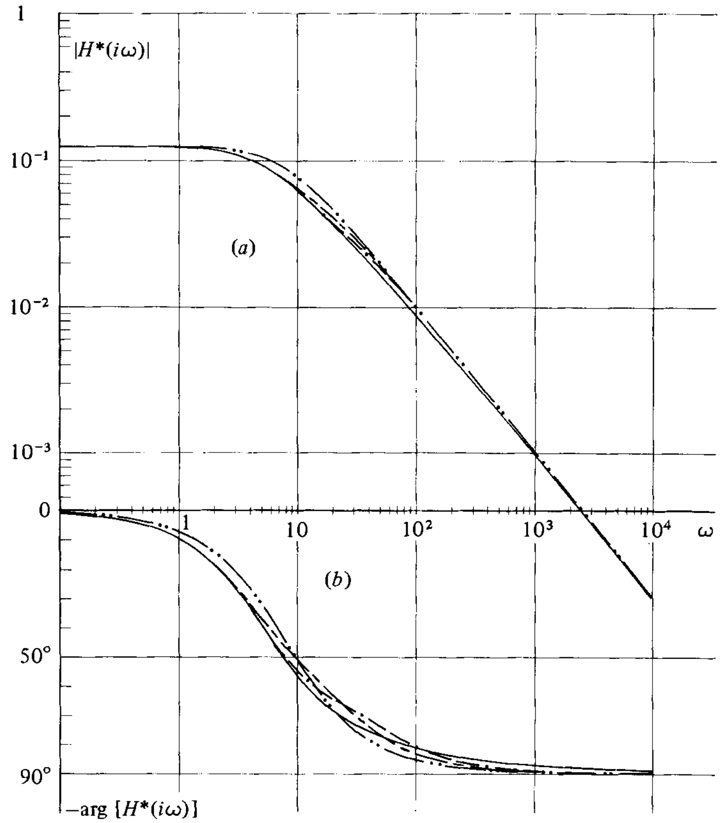

FIGURE 2. Modal approximation: (a) amplitude and (b) phase characteristics of the frequency response. ——, exact solution; — ·· — ··, quasi-steady approximation ($N = 0$); —— – ——, first-order modal approximation ($N = 1$); · — · —, second-order modal approximation ($N = 2$).

However, it must be stressed that an increase in N gives smaller and smaller improvements, and that high-order differential equations are not practically convenient. In some cases the high-frequency approximation we present in §2.4 will give better results and in a more convenient way.

2.4. *High-frequency approximation*

In the previous section, we have obtained the original $H(t)$ of $H^*(s)$ using the complex inversion formula. An alternative method consists in expanding $H^*(s)$ into a series of decreasing (fractional) powers of s and inverting this series term by term (Doetsh 1961). From the series expansion of $I_n(s)$ for large values of s, we deduce:

$$H_\tau^*(s) = \frac{2}{s^{\frac{1}{2}}} + \frac{3}{s} + \frac{15}{4s^{\frac{3}{2}}} + \dots, \tag{2.34}$$

which inverts into

$$H_\tau(t) = \frac{2}{\Gamma(\frac{1}{2})} t^{-\frac{1}{2}} + \frac{3}{\Gamma(1)} + \frac{15}{4\Gamma(\frac{3}{2})} t^{\frac{1}{2}} + \dots. \tag{2.35}$$

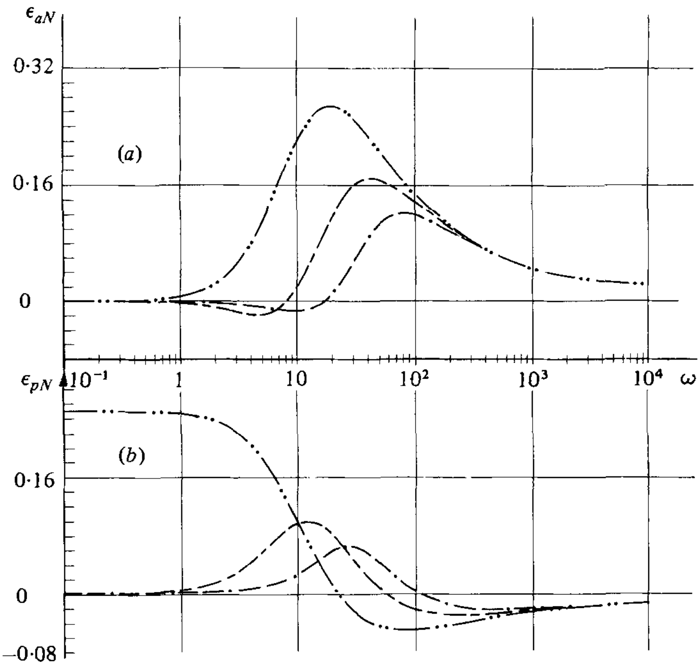

FIGURE 3. Modal approximation: relative errors in (*a*) amplitudes and (*b*) phases. — ·· — ··, quasi-steady approximation ($N = 0$); — – — –, first-order modal approximation ($N = 1$); — · — ·, second-order modal approximation ($N = 2$).

N	$\epsilon_{aN} < 5\%$	$\epsilon_{hN} < 5\%$
0	$\omega < 2{\cdot}8$	$\omega < 10^{-4}$
1	$\omega < 10{\cdot}1$	$\omega < 4{\cdot}2$
2	$\omega < 28$	$\omega < -18$

TABLE 3

This series converges for any $t \neq 0$, and its first term gives the asymptotic behaviour of H_τ for small values of t and high frequencies. Thus, according to (2.22), we can write the first-order approximation of the shear stress

$$\tau(t) = \frac{2}{\pi^{\frac{1}{2}}} \int_0^t \frac{1}{(t-t')^{\frac{1}{2}}} \frac{dW}{dt'} dt'. \tag{2.36}$$

It is interesting to notice that the same expression is obtained in a different way by Landau & Lifshitz (1971) for the shear stress on a flat plate oscillating in its own plane within a fluid originally at rest. This is fairly obvious since, for high frequencies, the diffusion of vorticity extends in a layer which remains thin compared to the pipe diameter. On the other hand, equation (2.36) is very similar to the expression of the transient viscous drag on a spherical particle in creeping motion (Basset force). These analogies suggest that the total shear stress could be written as the sum of a steady term and a convolution product describing the transient development of the boundary layer (Ishii & Chawla 1979).

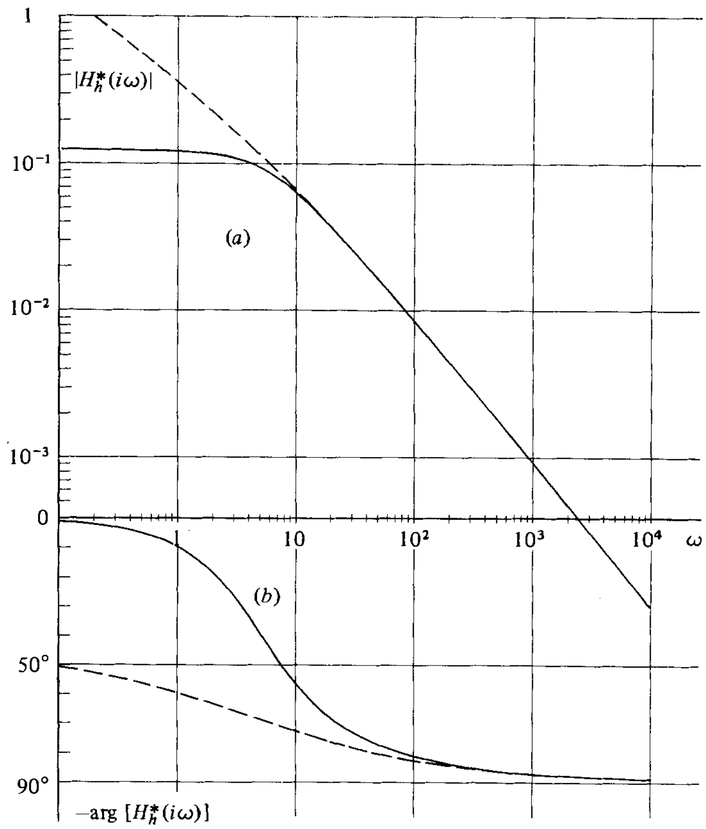

FIGURE 4. First-order high-frequency approximation: (a) amplitude and (b) phase characteristics of the frequency response compared with the exact solution. ——, exact solution; – – –, high-frequency approximation.

A detailed analysis of the frequency response can be made in the same way as for the modal approximation. Although successive approximations, involving an increasing number of terms, could be worked out easily, we shall limit ourselves to the first-order approximation, for which the general transfer function $H_h^*(s)$ can be written, according to (2.21),

$$H_h^*(s) = \frac{1}{s+2s^{\frac{1}{2}}}. \tag{2.37}$$

It is worth noticing that the zero-order approximation is no more than $H_{h0}^* = 1/s$, the transfer function of a frictionless system.

Figures 4 (a, b) give the comparison, within the range $\omega \in [10^{-1}, 10^4]$ of $|H_h^*(i\omega)|$ and the exact amplitude $|H^*(i\omega)|$, $\arg[H_h^*(i\omega)]$ and the exact phase $\arg[H^*(i\omega)]$ respectively.

Figure 5 represents the relative errors on amplitudes and phases

$$\epsilon_{ah} = \frac{|H_h^*(i\omega)| - |H^*(i\omega)|}{|H^*(i\omega)|}$$

and

$$\epsilon_{ph} = \frac{\arg[H_h^*(i\omega)] - \arg[H^*(i\omega)]}{\arg[H^*(i\omega)]}.$$

	Amplitudes	Relative error %	Phases	Relative error %
Exact value	0·03660	—	−68·73°	—
Second-order modal approximation	0·03724	+2	−64·68°	−6
First-order high-frequency approximation	0·03694	+1	−76·49°	+11

TABLE 4.

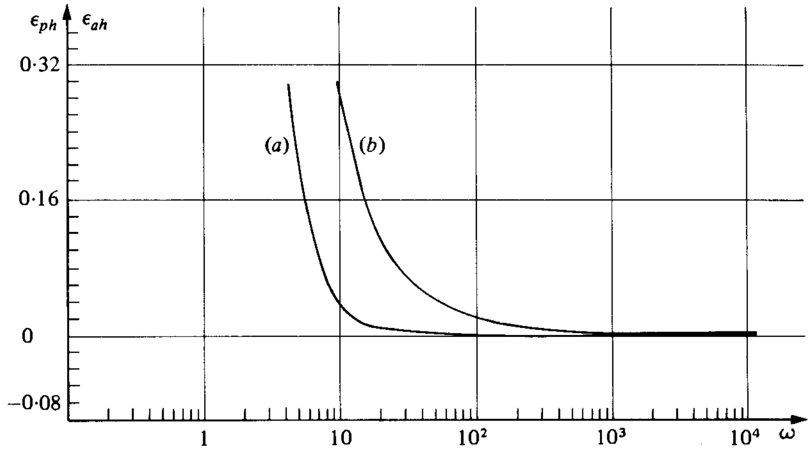

FIGURE 5. First-order high-frequency approximation: relative errors in (a) amplitudes and (b) phases.

From these results we observe that (i) the relative error ϵ_{ah} is less than 5 % for $\omega > 9$, which is surprisingly good; (ii) the relative error ϵ_{ph} is less than 5 % for $\omega > 39$; (iii) the approximation (2.36) is quite poor for low frequencies.

Practically, we conclude that amplitudes are reproduced, to the nearest ± 5 %, by the first-order modal approximation for $\omega \lesssim 10$ and by the first-order high-frequency approximation for $\omega \gtrsim 9$. But, as is well known, good accuracy is much more difficult to obtain for phases, a small gap remains between the second-order modal approximation ($\omega \leqslant 18$) and the first-order high-frequency approximation ($\omega \gtrsim 39$) if we keep the same accuracy of 5 %. So, either a second-order high-frequency approximation has to be developed or a slight compromise in accuracy is admitted. According to the values shown in table 4, we recommend that the separation between both approximations be chosen at $\omega = 20$.

3. An example of transient flows: the response to a step pressure gradient

The purpose of the present section is to give an example of the time behaviour of the average velocity $W(t)$ and the shear-stress $\tau(t)$, to compare the results of various approximations with the exact solution, and to make comments on some approximations which are proposed in the literature.

For a step pressure gradient, the exact solution is

$$W(t) = \tfrac{1}{8} - 4 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 t}}{\alpha_n^4}, \quad \tau(t) = \tfrac{1}{2} - 2 \sum_{n=1}^{\infty} \frac{e^{-\alpha_n^2 t}}{\alpha_n^2}. \qquad (3.1), (3.2)$$

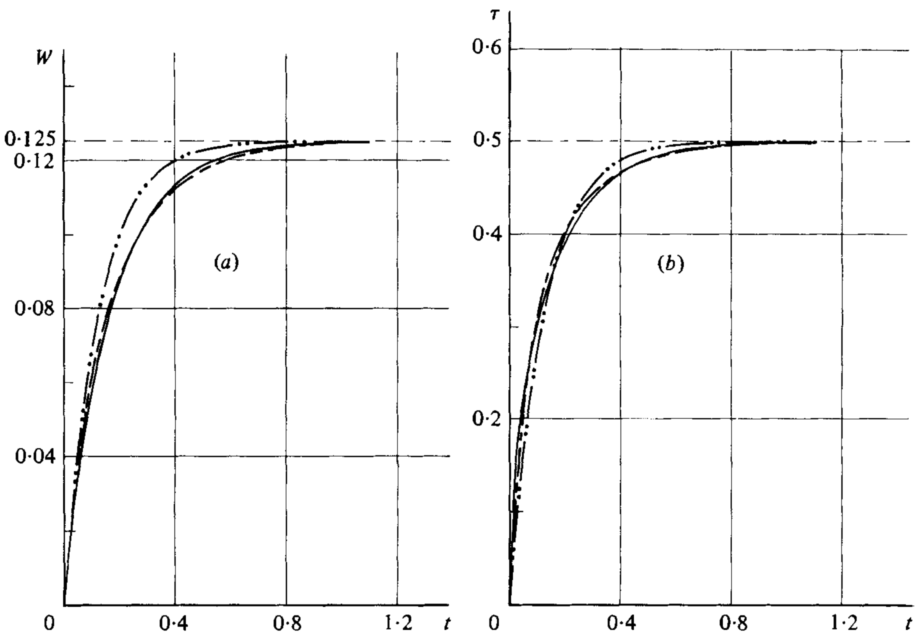

FIGURE 6. Step pressure gradient: time evolution of (a) velocity and (b) shear stress. ——, exact solution; ·· — ·· —, quasi-steady approximation; – ——– –, first-order modal approximation.

The chosen approximations are

(i) the quasi-steady approximation (3.26), which gives

$$W(t) = \tfrac{1}{8}(1-e^{-8t}), \quad \tau(t) = \tfrac{1}{2}(1-e^{-8t}); \qquad (3.3),\ (3.4)$$

(ii) the first-order modal approximation (2.29), the solution of which can be written

$$W(t) = \tfrac{1}{8} + Ee^{-Ct} + Fe^{-Dt}, \quad \tau(t) = \tfrac{1}{2} + E'e^{-Ct} + F'e^{-Dt}. \qquad (3.5),\ (3.6)$$

The coefficients in (3.5), (3.6) can be obtained from the four constants

$$A = 6+\frac{1}{2\sigma_1}\ (= 12{\cdot}02410), \quad B = \left(A^2-\frac{8}{\sigma_1}\right)^{\frac{1}{2}}\ (= 6{\cdot}94214),$$

$$C = A+B\ (= 18{\cdot}96624), \quad D = A-B\ (= 5{\cdot}08196)$$

by means of the following formulae

$$E = \frac{1-\sigma_1 C}{2\sigma_1 BC}, \quad F = \frac{\sigma_1 D-1}{2\sigma_1 BD},$$

$$E' = 4E-\frac{1}{B}, \quad F' = 4F+\frac{1}{B}.$$

Figures 6(*a*, *b*) compare the velocities and shear stresses respectively, and show very clearly the interest of the modal approximation.

Other empirical approximations have also been proposed in the literature: for instance Pham & Veteau (1977) write τ in terms of W:

$$\tau = 4W+0{\cdot}135\frac{dW}{dt}, \qquad (3.7)$$

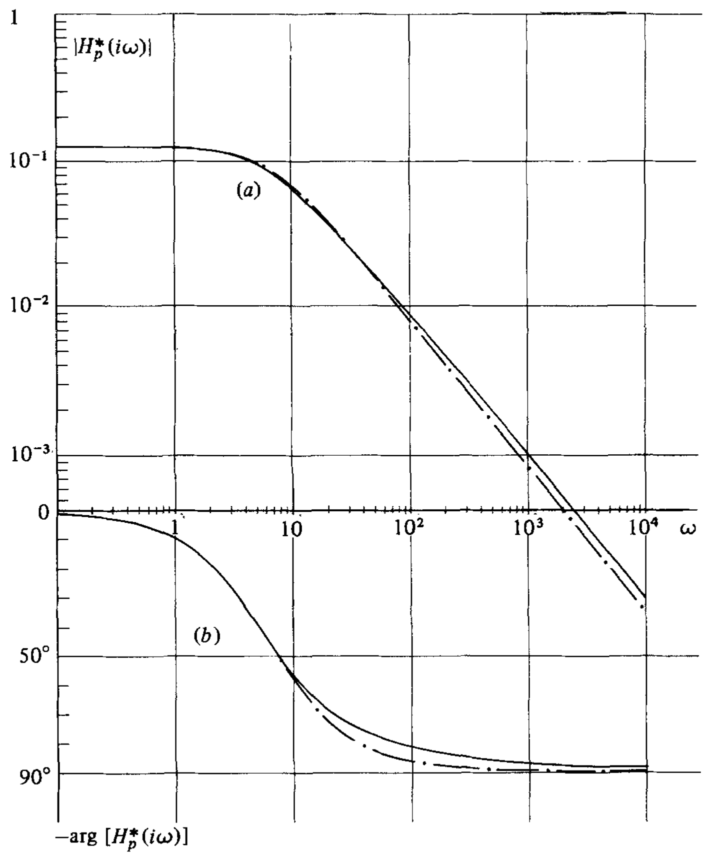

FIGURE 7. Empirical approximation (3.7): (*a*) amplitude and (*b*) phase characteristics of the frequency response showing the wrong behaviour at high frequencies. ——, exact solution; — · — ·, approximation (3.7).

where the numerical coefficient of the derivative has been calculated to fit the exact solution within the range of interest. Moreover, this approximation is the first-order representation of a Taylor-like expansion

$$\tau = 4W + \sum_{n=1}^{N} \delta_n \frac{d^n W}{dt^n}, \tag{3.8}$$

which appears, for instance, in the work of Letelier & Leutheusser (1978). Approximations of this kind seem extremely attractive, since they do not introduce in the calculations an extra differential equation: for instance, substituting (3.7) into the first-order differential equation (2.12) gives

$$(1+0{\cdot}270)\frac{dW}{dt} + 8W = G(t), \tag{3.9}$$

where the extra transient viscous term increases the inertia. The drawback of such an assumption is clearly illustrated by the transfer function which relates, in the image space of Laplace, the average velocity $W^*(s)$ to the pressure gradient $G^*(s)$:

$$H_p^*(s) = \frac{1}{8+1{\cdot}270s}. \tag{3.10}$$

Figures 7(*a*, *b*) represent the amplitude and phase characteristics of $H_p^*(s)$ compared with the exact transfer function, and they show that a good empirical choice of the parameter allows this approximation to give good results at low frequencies, but that, for higher frequencies, the results cannot be correct since the system tends towards a pure inertial behaviour with a wrong value of the inertia.

It is also important to notice that such an approximation has no physical meaning when $N > 1$. This comes from expression (2.20) which states, without any assumption, that, in the feedback branch of the schematic diagram (figure 1), the acceleration (or its image $sW^*(s)$) is the input and the shear stress is the output. Hence, any approximation of the transfer function $H_\tau^*(s)$ of this branch must comply with the elementary properties of a Laplace transform, and in particular tend towards zero when $s \rightarrow \infty$ (Takahashi, Rabins & Auslander 1972): this is clearly not verified for (3.8), the Laplace transform of which is

$$\tau^* = \frac{4}{s} + \sum_{n=1}^{N} \delta_n s W^*.$$

4. Conclusions

In this first part, we tried to establish a fully correct extension to any transient process of laminar pipe flow of the well-known steady wall shear-stress law which relates linearly the shear stress to the instantaneous average velocity.

A fundamental result due to Zielke (1968) was recalled in a simple way. Strictly speaking, the extra transient term which must be added to the steady-state relationship has to take into account the whole history of the velocity from the very origin of the motion. As the resulting convolution product is very cumbersome to handle in practical problems, two approximation procedures are presented.

The first one is valid for low-frequency cases, or large times in a transient process, and leads to a set of two ordinary, constant-coefficient, differential equations. In the closure equation, relating the shear stress to the velocity, relaxation terms appear *for the shear-stress as well as for the velocity*. The order of this differential equation must be increased according to the maximum frequency f_{max} involved in the process; but a complete study of the accuracies shows that the simple first-order differential equation (3.29) gives excellent results up to $f_{max} \sim 10\nu/2\pi R^2$ and that the second-order equation (3.30) is valid until $f_{max} \sim 20\nu/2\pi R^2$. Moreover, these modal approximations remain physically consistent even for large frequencies, since they allow the system to have a purely inertial behaviour when $f \rightarrow \infty$.

The second procedure is concerned with high-frequency cases only, or the very first motion in the case of a fast transient. Since it comes from an asymptotic expansion when $f \rightarrow \infty$, this approximation should by no means be applied to low-frequency cases. However, it gives good results for $f_{min} > 20\nu/2\pi R^2$ and satisfactory ones down to $f_{min} > 10\nu/2\pi R^2$. The convolution product (2.36) expressing the shear stress in terms of the velocity is much simpler than in the general case. Although it is not impossible to approximate this integral by a differential relationship, the coefficients are no longer constants but involve successive powers of $t^{-\frac{1}{2}}$, and no attempt was made to calculate the limit value of the numerical constants. The reason being that the practical importance of the high-frequency approximation is much less than in the previous case, since it is useless in most transient processes and restricted to periodic motions with a fundamental frequency larger than the limit frequency $f_{\min}$.

The present study was limited to laminar flows and circular cylindrical pipes. A first extension to other cylindrical geometries can be done without any new fundamental problem whenever a set of eigenfunctions is available for the exact solution (in rectangular pipes for instance); but, should the computation be impossible, the relaxation coefficients appearing in (2.29) or (2.30) could easily be obtained from a typical experiment. Clearly, a more interesting, and more difficult, extension would concern turbulent flows. It can be anticipated that two different ways could be followed.

(*a*) A simplified model could be used which would be analytically tractable. Although this direction does not seem to be very promising, for unstationary turbulent models are neither numerous nor reliable, an interesting attempt is reported by Ishii & Chawla (1979) who use a 'penetration model' to establish a convolution-type transient correction of the shear stress.

(*b*) It could be postulated that the structure of the wall-shear stress law, stated for laminar flows only, can be extended to turbulent cases. In fact, it is likely that the form of the low-frequency differential relationship is fairly general, and that the relaxation coefficients could be obtained from careful experiments. It must be stressed, however, that the linearity of this relationship would limit an identification attempt to small-amplitude pressure steps or oscillations around a given stationary flow.

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